Names Key Class ________ Date ________

5.5 Other Trig Formulas Rally Coach

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u \qquad \cos 2u = \cos^2 u - \sin^2 u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u} \qquad = 2\cos^2 u - 1$$

$$= 1 - 2\sin^2 u$$

These formulas can be easily derived from the "Sum and Difference" formulas.

Note: There are 3 different double angle formulas for cosine. Use the one that is most convenient.

EXAMPLE 1 **Solving a Multiple-Angle Equation**

Solve $2 \cos x + \sin 2x = 0$.

Solution Begin by rewriting the equation so that it involves functions of x (rather than $2x$). Then factor and solve.

$2 \cos x + \sin 2x = 0$	Write original equation.
$2 \cos x + 2 \sin x \cos x = 0$	Double-angle formula
$2 \cos x(1 + \sin x) = 0$	Factor.
$2 \cos x = 0$ and $1 + \sin x = 0$	Set factors equal to zero.
$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad x = \frac{3\pi}{2}$	Solutions in $[0, 2\pi)$

So, the general solution is

$$x = \frac{\pi}{2} + 2n\pi \quad \text{and} \quad x = \frac{3\pi}{2} + 2n\pi$$

Solve each of the following equations over the interval [0, 2π).

1a.) *cos(2x) + cosx = 0*

See paper

$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

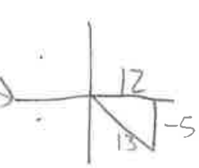
1b.) *tan(2x) – cotx = 0*

See paper

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Use the given information to find the requested trig ratio for the double angle. (Hint: Drawing a diagram may be helpful.)

2a.) If $\sin\theta = -\frac{5}{13}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find the value of $\sin(2\theta)$.



$= 2 \sin\theta \cos\theta$

$= 2\left(\frac{-5}{13}\right)\left(\frac{12}{13}\right)$

$= \frac{-120}{169}$

2b.) If $\sin\theta = -\frac{5}{13}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find the value of $\cos(2\theta)$.

$= 1 - 2\sin^2\theta$

$= 1 - 2\left(\frac{-5}{13}\right)^2$

$= 1 - 2\left(\frac{25}{169}\right)$

$= 1 - \frac{50}{169} = \frac{169}{169} - \frac{50}{169} = \frac{119}{169}$

Power Reducing Formulas:

$$\sin^2 u = \frac{1-\cos 2u}{2} \qquad \cos^2 u = \frac{1+\cos 2u}{2} \qquad \tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$$

These formulas are useful when trying to re-write a trig expression in a form without exponents. This is helpful in calculus.

Example 2: Rewrite $\sin^4 x$ in terms of first powers of the cosines of multiple angles.

Solution Note the repeated use of power-reducing formulas.

$\sin^4 x = (\sin^2 x)^2$	Property of exponents
$= \left(\frac{1-\cos 2x}{2}\right)^2$	Power-reducing formula
$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$	Expand.
$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)$	Power-reducing formula
$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x$	Distributive Property
$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$	Simplify.
$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$	Factor out common factor.

Use the power-reducing formulas to re-write each of the following expressions in terms of the first power of cosine.

3a.) *$\cos^4 x$*

$(\cos^2 x)^2$

$\left(\frac{1+\cos 2x}{2}\right)^2$

$\frac{1+2\cos 2x+(\cos^2 2x)}{4}$

$\frac{1+2\cos 2x+\left(\frac{1+\cos 4x}{2}\right)}{4}$

3b.) *$\tan^4(2x)$* *Note the double angle.*

See paper for 3b

$\frac{2}{2}\left(\frac{1+2\cos 2x+\left(\frac{1+\cos 4x}{2}\right)}{4}\right)$

$\frac{2+4\cos 2x+1+\cos 4x}{8} = \frac{3+4\cos 2x+\cos 4x}{8}$

You can derive some useful alternative forms of the power-reducing formulas by replacing u with $u/2$. The results are called **half-angle formulas.**

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The signs of $\sin\frac{u}{2}$ and $\cos\frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.

Example 3: Find the exact value of $\sin 105°$.

Solution Begin by noting that 105° is half of 210°. Then, using the half-angle formula for $\sin(u/2)$ and the fact that 105° lies in Quadrant II, you have

$$\sin 105° = \sqrt{\frac{1-\cos 210°}{2}} = \sqrt{\frac{1+(\sqrt{3}/2)}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

The positive square root is chosen because $\sin\theta$ is positive in Quadrant II.

Use the Half-Angle formulas to find the exact value of each trig function for the given angle.

4a.) $\cos\frac{\pi}{8} = \left(\frac{\frac{\pi}{4}}{2}\right)$

$\cos\left(\frac{\frac{\pi}{4}}{2}\right) = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}$

$= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}$

$= \sqrt{\frac{1}{2}+\frac{\sqrt{2}}{4}}$ or $\sqrt{\frac{2+\sqrt{2}}{4}}$ or $\boxed{\frac{\sqrt{2+\sqrt{2}}}{2}}$

4b.) *tan75°* $= \tan\left(\frac{150}{2}\right) = \frac{1-\cos(150)}{\sin(150)}$

$= \frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}}$

$= \frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$= 2+\sqrt{3}$

Use the half-angle formulas to find all of the solutions to the equation over the interval [0, 2π).

5a.) $\cos^2 x = \sin^2\frac{x}{2}$

$\cos^2 x = \left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2$

$2\left(\cos^2 x = \frac{1-\cos x}{2}\right)$

$2\cos^2 x = 1-\cos x$

$2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\cos x = \frac{1}{2}$ $\quad$ $\cos x = -1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\quad$ $x = \pi$ ✓

5b.) $\sin\frac{x}{2} + \cos x = 0$

$\sin\frac{x}{2} = -\cos x$

$\left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2 = (-\cos x)^2$

$2\left(\frac{1-\cos x}{2} = \cos^2 x\right)$

$1-\cos x = 2\cos^2 x$

$0 = 2\cos^2 x + \cos x - 1$

$0 = (2\cos x - 1)(\cos x + 1)$

$x = \cancel{\frac{\pi}{3}}, \cancel{\frac{5\pi}{3}}, \pi$

check for extraneous!

$\sin\left(\frac{\frac{\pi}{3}}{2}\right) + \cos\frac{\pi}{3} = 0$

$\sin\frac{\pi}{6} + \cos\frac{\pi}{3} = 0$

$\frac{1}{2} + \frac{1}{2} \neq 0$

$\sin\left(\frac{5\pi}{6}\right) + \cos\frac{5\pi}{3} = 0$

$\frac{1}{2} + \frac{1}{2} \neq 0$

$\sin\frac{\pi}{2} + \cos\pi = 0$

$1 + -1 = 0$ ✓

$x = \pi$

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)] \qquad \sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)] \qquad \cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$$

These formulas are used in calculus to solve problems involving the products of sines and cosines of different angles.

Rewrite the product as a sum or difference using the Product-to-Sum formulas.

6a.) *cos(5x)sin(4x)*

$\frac{1}{2}\left[\sin(9x) - \sin(x)\right]$

6b.) *sin(5x)sin(3x)*

$\frac{1}{2}\left[\cos(2x) - \cos(8x)\right]$

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \qquad \cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \qquad \cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

These formulas are helpful if you have the sum (or difference) of a trig function with 2 different angles. The formulas allow you to rewrite the sum/difference as a product and then use the zero product property.

Example 4: G

Solve $\sin 5x + \sin 3x = 0$.

Solution

$\sin 5x + \sin 3x = 0$ Write original equation.

$2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right) = 0$ Sum-to-product formula

$2\sin 4x \cos x = 0$ Simplify.

2sin $4x$ = 0 yields (in the interval [0, 2π))

cos x = 0 yeilds

$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ which are repeat solutions

Final answer: $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

Find the exact value of the following expression by using the appropriate sum-to-product formula.

7a.) *cos 195° + cos 105°*

$2\cos\left(\frac{300}{2}\right)\cos\left(\frac{90}{2}\right)$

$2\cos(150)\cos(45)$

$\cancel{2}\left(-\frac{\sqrt{3}}{\cancel{2}}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{2}$

7b.) *sin 195° + sin 105°*

$2\left(\sin\frac{300}{2}\right)\cdot\left(\cos\frac{90}{2}\right)$

$2\sin(150)\cdot\cos(45)$

$2\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{2}}{2}$

Solve the following equation over the interval [0, 2π).

8a.) *sin 5x + sin 3x = 0*

$2\sin\left(\frac{8x}{2}\right)\cdot\cos\left(\frac{2x}{2}\right) = 0$

$2\cdot\sin 4x\cdot\cos x = 0$

$2 \neq 0$ $\quad \sin 4x = 0$ $\quad \cos x = 0$

$\sin(u) = 0$ $\quad x = \frac{\pi}{2}, \frac{3\pi}{2}$

$u = \pi n$

$\frac{4x}{4} = \frac{\pi}{4}n$

$x = \frac{\pi}{4}n$

Repeat solutions

$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

8b.) *cos 2x – cos 6x = 0*

$-2(\sin 4x)(\sin -2x) = 0$

$\sin 4x = 0$ See Previous Question

$\sin -2x = 0$

$\sin u = 0$

$u = \pi n$

$-2x = \pi n$

$x = -\frac{\pi}{2}n$ $\quad n = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

gives same angles

$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

1a) $\cos(2x) + \cos x = 0$

Sub. $(2\cos^2 x - 1) + \cos x = 0$

$2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$2\cos x - 1 = 0$ $\qquad$ $\cos x + 1 = 0$

$2\cos x = 1$ $\qquad$ $\cos x = -1$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\qquad$ $x = \pi$

$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

1b.) $\tan(2x) - \cot x = 0$

$\frac{(\tan x)}{(\tan x)} \frac{2\tan x}{1-\tan^2 x} - \frac{1}{\tan x} \frac{(1-\tan^2 x)}{(1-\tan x)} = 0$

$$\frac{2\tan^2 x - (1 - \tan^2 x)}{\tan x\,(1-\tan^2 x)} = 0$$

$$(\tan x)(1-\tan^2 x)\,\frac{2\tan^2 x + \tan^2 x - 1}{\cancel{\tan x\,(1-\tan^2 x)}} = 0\,(\tan x)(1-\tan^2 x)$$

$3\tan^2 x - 1 = 0$

$3\tan^2 x = 1$

$\tan^2 x = \frac{1}{3}$

$\tan x = \pm\frac{1}{\sqrt{3}}$ Think $\frac{1}{2} = \sin$

$\frac{\sqrt{3}}{2} = \cos$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

3b.) $\tan^4(2x)$

$$= \left(\tan^2(2x)\right)^2$$

$$= \left(\frac{1-\cos 2(2x)}{1+\cos 2(2x)}\right)^2$$

$$= \left(\frac{1-\cos 4x}{1+\cos 4x}\right)^2$$

$$= \frac{1-2\cos 4x + (\cos^2 4x)}{1+2\cos 4x + (\cos^2 4x)}$$

$$=_2 \frac{\left(1-2\cos 4x + \frac{1+2\cos 8x}{2}\right)}{2\left(1+2\cos 4x + \frac{1+2\cos 8x}{2}\right)}$$

$$\frac{2-4\cos 4x + 1 + 2\cos 8x}{2+4\cos 4x + 1 + 2\cos 8x} = \boxed{\frac{3-4\cos 4x + 2\cos 8x}{3+4\cos 4x + 2\cos 8x}}$$