Learn

In the following activity, you will be shown a list of numbers separated by commas, which is known as a sequence. Your objective is to identify the pattern within each sequence and to use that pattern to determine the next numbers, or terms, in the list.

Find the next three terms in the sequence.

Type your answers in the spaces provided. If the answer is a fraction, enter the decimal equivalent.

8, 23, 38, 53, ___, ___, ___, …

−5, −7, −9, −11, ___, ___, ___, …

3, 1.75, 0.5, −0.75, ___, ___, ___, …

Notice that the terms within each sequence were separated by a common difference. In the list 8, 23, 38, 53 …, the terms 8 and 23, 23 and 38, 38 and 53 all have a common difference of positive 15. As a result, each list is an example of an arithmetic sequence.

Essential Questions

After completing this lesson, you will be able to answer the following questions:

- How can an expression or process be determined for an arithmetic sequence?
- What functions combine to create an explicit formula for arithmetic sequences?
- What possible restrictions exist on domains and ranges of arithmetic sequences?
- How can an expression or process be determined for an arithmetic series?

Arithmetic Sequences

An arithmetic sequence is a list of numbers, called terms, which share a common difference.

Different Forms

A sequence can appear in multiple forms. The most common is a list of terms; however, other ways to display the data, like a table or a graph, may also be used. Here is the same sequence represented three different ways:

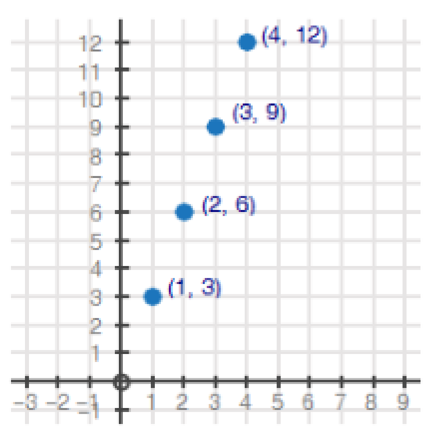
List of Terms

3, 6, 9, 12, …

Table

n	f(n)
1	3
2	6
3	9
4	12

Graph



Important Note

You will notice on the graph that there is not a continuous line through the points. That is because a sequence is a discrete set of points. This means that the sequence only exists at those points. That is different from a graph of a line which is a continuous set of points.

Recursive Versus Explicit

There are two different methods for determining the terms of a sequence.

Recursive

A recursive process requires the use of the previous term in a sequence. "Add two to get the next number" is what you might say when thinking recursively. A recursive process could be listed as $a_n = a_{n-1} + 2$. The subscript n and n − 1 refer to the place in the sequence a term is. The first term is a_1, the second term is a_2, and so on.

Using the recursive process $a_n = a_{n-1} + 2$ we can find the terms of an arithmetic sequence given the first term. Assume the first term, a_1, equals 3.

$$a_1 = 3$$
$$a_2 = a_{2-1} + 2 = a_1 + 2 = 3 + 2 = 5$$
$$a_3 = a_{3-1} + 2 = a_2 + 2 = 5 + 2 = 7$$
$$a_4 = a_{4-1} + 2 = a_3 + 2 = 7 + 2 = 9$$

The word "recursive" contains the prefix "re-", which means "again." A recursive process is where each term must be known and the process is applied again and again.

Explicit

An explicit formula allows you to calculate any term in a sequence directly, unlike a recursive process that requires the term before it be identified first.

An explicit formula is useful in ways the recursive process would be tiresome. Imagine if you were asked to find the 45th term of an arithmetic sequence, such as 2, 9, 16, …? Sure, it's possible to list each term until you reach the 45th entry. But that would take forever! This is where the explicit formula can help.

Deriving the Explicit Formula

1st Term		2
2nd Term	2 + (1)(7)	9
3rd Term	2 + (2)(7)	16
4th Term	2 + (3)(7)	23

You know the first term is 2.

The second term is found by adding 7 to the first term.

The third term is found by adding 7 to the second term or by adding 14, which is 7 + 7, to the first term.

The fourth term is found by adding 7 to the third term or by adding 21, which is 7 + 7 + 7, to the first term.

Using the pattern shown above, where does the 2 originate?

2 is the first term of the sequence.

Where does the 7 originate?

7 is the common difference between each term of the arithmetic sequence.

How is the factor before the 7 determined?

The factor before the common difference comes from the position of the term in the sequence minus 1.

What is the fifth term of the sequence?

30

Arithmetic Sequence Formula:

$$a_n = a_1 + (n - 1)d$$

In this formula, a_1 represents the first term of the sequence, n is the position of the term in the sequence, d is the common difference, and an is the desired term of the sequence.

Applying this formula to find the 25th term of the arithmetic sequence 2, 9, 16 … will result in the following.

$$a_n = a_1 + (n - 1)d$$

$$a_{25} = 2 + (25 - 1)(7)$$

$$a_{25} = 2 + (24)(7)$$

$$a_{25} = 2 + (168)$$

$$a_{25} = 170$$

The 25th term of the arithmetic sequence 2, 9, 16 … is 170. That was much easier than listing all 25 terms of the sequence!

Converting Recursive and Explicit Equations

For a given arithmetic sequence, the recursive and explicit formulas are as follows:

Recursive	Explicit
$a_n = a_{n-1} + d$	$a_n = a_1 + d(n - 1)$

Notice where the expressions a_n, **n − 1**, and **d** are in both equations. Knowing where to place the different expressions will allow you to utilize either formula.

Domain and Range

In math, the nature of the situation can put restrictions on acceptable domains and ranges. This is very true in real-world situations because you would not want solutions resulting in things like half footballs or negative lengths. The same is true for arithmetic sequences.

The Domain

A sequence is a set of numbers, and we count the terms as the first term, the second term, and so on. It would not be appropriate to have a negative term, a decimal term, or a fraction term. Nope, the terms start at 1!

The domain for arithmetic sequences is then restricted to all integers greater than or equal to 1. These are also sometimes referred to as "natural numbers" because they are the numbers you naturally learn when you first learned to count.

Also remember that in the explicit formula, the expression n − 1 is used. Any integer less than one would result in a negative number and would be outside of the sequence.

The Range

The range can be any number. It does not have the same restrictions that the domain has. Freedom!

Example

Identify the 46th term of an arithmetic sequence where $a_1 = -22$ and $a_{12} = 77$.

Find d

To find the 46th term of this sequence, you will first need to know the common difference between the terms.

In this arithmetic sequence, you are given the first term ($a_1 = -22$), the 12th term ($a_{12} = 77$), and the position of the desired term (46). As a result, you can't directly find the common difference. Instead, apply the arithmetic sequence formula using the 12th term as the last term of a shorter sequence, and then find the 46th term.

$$a_n = a_1 + (n - 1)d$$

$$a_{12} = a_1 + (12 - 1)d$$

Substitute the values of the first and last terms of the sequence into the formula and solve for the common difference.

$$77 = -22 + (12 - 1)d$$

$$77 = -22 + (11)d$$

$$+22 \quad +22$$

$$99 = 11d$$

$$9 = d$$

Find a_{46}

Now that you know the common difference, you can use this value to find the 46th term of the sequence.

$$a_n = a_1 + (n - 1)d$$

$$a_{46} = -22 + (46 - 1)(9)$$

$$a_{46} = -22 + (45)(9)$$

$$a_{46} = -22 + (405)$$

$$a_{46} = 383$$

The 46^{th} term of the arithmetic sequence is 383.

Example

Identify the 29^{th} term of the arithmetic sequence 11, 8, 5, 2…

To find the 29^{th} term of this arithmetic sequence, you will need the formula $a_n = a_1 + (n - 1)d$. In order to use this formula, the value for all but one of the variables must be defined.

The first term of the sequence is 11, so $a_1 = 11$. Since you are looking for the 29^{th} term, n will equal 29. The common difference between each term of the sequence is −3.

−3 −3 −3

11, 8, 5, 2…

Substitute each of these values into the formula. Solve for the missing variable.

$$a_n = a_1 + (n - 1)d$$

$$a_{29} = 11 + (29 - 1)(-3)$$

$$a_{29} = 11 + (28)(-3)$$

$$a_{29} = 11 + (-84)$$

$$a_{29} = -73$$

Arithmetic Series

An arithmetic series, similar to an arithmetic sequence, is a list of numbers, called terms, which share a common difference. However, the terms in an arithmetic series are being added rather than separated by commas.

Arithmetic Series Formula:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

In the arithmetic series formula, the variable S_n represents the sum of a series.

Find the sum of the arithmetic series 4 + 1 + −2 + −5 + −8 + −11 by applying the formula.

There are six terms in this series **($n = 6$)**, where the first term is **4($a_1 = 4$)**, and the last term is **−11($a_n = -11$)**. Substitute these values in the formula to find the sum of the series.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_6 = \frac{6}{2}(4 - 11)$$

$$S_6 = \frac{6}{2}(-7)$$

$$S_6 = 3(-7)$$

$$S_6 = -21$$

However, as you have learned, sometimes the arithmetic sequence or series is too long to list all of the terms. In these situations, you might not know the value of the last term. In these cases, a second formula, using the common difference instead of the last term, can be used.

Recall that the last term of an arithmetic sequence can be found by the formula $a_n = a_1 + (n - 1)d$. By substituting the expression $a_1 + (n - 1)d$ for a_n in the arithmetic series formula and simplifying, you can find the other formula!

Arithmetic Series Formula:

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

Practice

What is the sum of a 44-term arithmetic sequence where the first term is −9 and the last term is 120?

The sum of a 44 term arithmetic sequence where the first term is −9 and the last term is 10 is 2,442.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{44} = \frac{44}{2}(-9 + 120)$$

$$S_{44} = \frac{44}{2}(111)$$

$$S_{44} = 22(111)$$

$$S_{44} = 2{,}442$$

07.01 Lesson Summary

To achieve mastery of this lesson, make sure that you develop responses to the essential questions listed below:

- How can an expression or process be determined for an arithmetic sequence?
- What types of functions combine to create an explicit formula for arithmetic sequences?
- What possible restrictions exist on domains and ranges of arithmetic sequences?
- How can an expression or process be determined for an arithmetic series?

Arithmetic Sequences

A sequence is a list of numbers, also called terms, separated by commas. An arithmetic sequence is a list of numbers where the difference between each term is constant.

A specified term of an arithmetic sequence may be found by the formula $a_n = a_1 + (n - 1)d$ where a_1 is the first term of the sequence, n is the position number of a given term, d is the common difference between the terms, and a_n is the value of the term in the given position number.

Arithmetic Sequence Formula:

$$a_n = a_1 + (n - 1)d$$

Recursive vs. Explicit

- A recursive process requires the calculation of each term as each term relies on the term before it.
- An explicit equation allows the calculation of any term in the sequence.

Recursive	Explicit
$a_n = a_{n-1} + d$	$a_n = a_1 + d(n - 1)$

Restrictions

- The domain is restricted to the integers greater than or equal to 1. These are the counting or “natural” numbers.
- The range is unrestricted.

Arithmetic Series

An arithmetic series is a list of numbers separated by a common difference that are being added.

The sum of an arithmetic series may be found by the formula $S_n = \frac{n}{2}(a_1 + a_n)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ where a_1 is the first term of the series, n is the number of terms in the series, d is the common difference between the terms, a_n is the final term in the series, and S_n is the sum of the series.

Arithmetic Series Formula:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

Exam: 07.01 Arithmetic Sequences and Series

Question 1(Multiple Choice Worth 2 points)

(07.01)

Identify the 27th term of an arithmetic sequence where $a_1 = 38$ and $a_{17} = -74$.

- −20.5
- −151
- −22.75
- −144

Question 2(Multiple Choice Worth 2 points)

(07.01)

Given an arithmetic sequence in the table below, create the explicit formula and list any restrictions to the domain.

n	a_n
1	9
2	3
3	−3

- $a_n = 9 - 3(n - 1)$ where $n \le 9$
- $a_n = 9 - 3(n - 1)$ where $n \ge 1$
- $a_n = 9 - 6(n - 1)$ where $n \le 9$
- $a_n = 9 - 6(n - 1)$ where $n \ge 1$

Question 3(Multiple Choice Worth 2 points)

(07.01)

The seats at a local baseball stadium are arranged so that each row has 5 more seats than the row below it. If there are 4 seats in the 1st row, how many seats are in row 23?

- 110
- 114
- 115
- 119

Question 4(Multiple Choice Worth 2 points)

(07.01)

Given the functions f(n) = 25 and g(n) = 3(n − 1), combine them to create an arithmetic sequence, a_n, and solve for the 12th term.

- $a_n = 25 - 3(n - 1)$; $a_{12} = -11$
- $a_n = 25 - 3(n - 1)$; $a_{12} = -8$
- $a_n = 25 + 3(n - 1)$; $a_{12} = 58$
- $a_n = 25 + 3(n - 1)$; $a_{12} = 61$

Question 5(Multiple Choice Worth 2 points)

(07.01)

Identify the 42nd term of an arithmetic sequence where $a_1 = -12$ and $a_{27} = 66$.

- 70
- 72
- 111
- 114